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MODELLING OF A WIRELESS CHANNEL WITH TIME DEPENDENT PARAMETERS

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Abstract: It is convenient to analyse wireless channels and links by exploiting their input-output description. This approach relies on treating the system as a black box, whose behaviour can be fully described by the relationship between the input and output signals. In this paper, we study a relationship of the above type for linear wireless channels having time-dependent parameters, which also takes a multipath propagation environment into account. A starting point for the derivations presented here is a relationship derived in the literature for this type of model with the application of a single sinusoidal input signal. The subject of this paper is the generalization of that relationship for periodic input signals and then of non-periodic signals. To the best of our knowledge, the literature lacks a suitably convincing generalization. The derivations of this paper exploit a principle of superposition valid for linear systems as well as the relations existing between Fourier series and the Fourier integrals. The discussion is illustrated by the results of simulations performed with the help of the MATLAB program.

Keywords: wireless channel, superposition principle, multipath, Fourier transform.

1. INTRODUCTION

Customarily, models of the radio channels and links used in wireless system analyses are based on a relation that links the output and input signals. This type of description is referred to as an input-output description, and the resulting model as a black box model.

In [Tse and Viswanath 2005] an entire section is devoted to the modelling of wireless channels under the convention outlined above. It demonstrates in an easy to understand manner, without resorting to solving Maxwell equations, how to obtain useful and practical models of radio channels for various configurations of beneficial/adverse effects of the propagation phenomena (wave reflection, refraction, attenuation, diffraction). The effects of various obstacle configurations are analysed in the form of the resulting waves reaching the receiver. These comprise waves that are reflected from various obstacles that reach the receiver via multiple paths, i.e. with different delays. Such a channel is said to be multipath. In [Tse and Viswanath 2005], an equation is derived, characterising this type of channel based on a single sinusoidal signal $x(t) = \cos(2\pi ft)$ input to the channel. This equation is as follows:

$$y(t) = \sum_i a_i(f,t)\cos 2\pi f(t - \tau_i(f,t)), \tag{1}$$

where $y(t)$ is the output signal of the channel, i.e. the received signal; i is the index of the consecutive constituent wave reaching the receiver, i.e. along the path with the index i; $a_i(f, t)$ is wave amplitude attenuation and $\tau_i(f, t)$ is the time delay on path i. Additionally in (1), t is time and f is frequency.

As was noted above, the correctness of equation (1) is demonstrated in [Tse and Viswanath 2005] only for the situation where the transmitted signal is a sinusoidal signal, i.e. $x(t) = \cos(2\pi ft)$. For signals with other forms, it is only noted that if the relations between $a_i(f, t)$ and $\tau_i(f, t)$ parameters and frequency are ignored then the superposition principle allows formula (1) to be as follows:

$$y(t) = \sum_i a_i(t)x(t - \tau_i(t)), \tag{2}$$

where $x(t)$ is now any input signal. The purpose of the current paper is to demonstrate, step by step, the validity of formula (2).

The superposition principle [Nurfaizey et al. 2012], as it relates to electrical and magnetic fields, states that if multiple field sources exist, the resulting field in a specific point of space is a vector sum of the fields originating from these sources - treated as individual sources, while for waves it is the algebraic sum of the individual disturbances. In the case of radio channels, the field in question is an electromagnetic field, described using Maxwell equations. It is also known that in a far field [Tse and Viswanath 2005], the electromagnetic field can be uniquely defined by providing only one of its constituents, e.g. electric field intensity. As noted above, this intensity can be determined by solving the above-mentioned Maxwell equations. In [Tse and Viswanath 2005], an example solution to these equations is quoted when stimulating the channel with a $x(t) = \cos(2\pi ft)$ signal, with the following formula being given:

$$E_r(f,t,\boldsymbol{u}) = \frac{\alpha(\theta,\psi,f)\cos 2\pi f(t - r/c)}{r}, \tag{3}$$

for the polar constituent r of electric field intensity $Er(.)$ at point $\boldsymbol{u}$ (located in the far field), defined by vector $\boldsymbol{u}$ with polar coordinates (θ,ψ,r). Coefficient $\alpha(.)$ defines the combined radiation characteristics of the transmitting and receiving antennas, while c is the speed of light in a vacuum. For convenience, this paper, as did [Tse and Viswanath 2005], identifies the channel response denoted in formulas (1) and (2) as $y(t)$, with electric field intensity $E_r(.)$ as in formula (3), or with the resultant electric field intensity at a specific point in space (an equivalent of equation (2)).

The above means that if a complex input signal meets the Dirichlet conditions and can be represented as a sum of sinusoidal signals ($x_k(t) = \cos(2\pi f_k t)$ and $x_k(t) = \sin(2\pi f_k t) = \cos(2\pi f_k t - \pi/2)$) at the frequencies f_k, $k = 1,2,3,\ldots$, then using the superposition principle, formula (3) can be applied to the individual components of this sum. The resultant electric field intensity value is then equal to the sum of the constituents taking the form of (3), where frequency f consecutively takes the values of f_k, $k = 1,2,3,\ldots$. The pattern outlined above will be used in the derivations discussed in the following sections, although to obtain a decomposition of the periodic signal as a sum of individual sinusoidal signals, a Fourier transform [Szabatin 2016], which is obvious in this case, will be used. The situation is much more difficult for non-periodic signals, but the literature provides certain methods (proposed in, for example [Landman et al. 2008]), as used here.

The remainder of the paper is organised as follows. Section 2 provides a derivation of formula (2) for periodic input signals, while Section 3 discusses a representation of non-periodic signals, which is then used to derive the channel input-output relations for these signals in Section 4. The paper concludes with a brief summary.

2. DERIVATION OF INPUT-OUTPUT RELATIONS FOR PERIODIC SIGNALS

As noted above, in accordance with the superposition principle, the value of electric field intensity at a given point in space equals the sum of the responses originating from individual sources. This paper analyses a radio channel between a single transmitting antenna and a single receiving antenna, which means a scenario where only a single radio signal source is present in the space. This means it can be treated as a point source [Tse and Viswanath 2005]. If the signal from this source is a periodic signal describable by a Fourier series, then its components (individual sinusoidal signals) can be treated as separate sources (emitting from the same point in space). This is the method used in this paper. It is also assumed that the above sources generate single sinusoidal signals reflected from obstacles in the communication path, which causes the occurrence of the multipath phenomenon [Tse and Viswanath 2005] and attenuation, before being ultimately summed in accordance with the superposition principle in the receiver with a certain delay. It is assumed that this occurs with attenuations $ai(t)$ and delays $\tau_i(t)$ along the individual paths I, independent of the frequency [Tse and Viswanath 2005].

Note that formula (1) can then be applied to individual sinusoidal signals (components of the Fourier series for the periodic signal) received in the receiver over the multipath radio channel, characterised by the presence of summation $\sum_i$ in formula (1). In this case, the variable f present in formula (1) takes the subsequent values of the periodic signal basic frequency.

These individual constituents received in the receiver, defined above, are denoted as: $y_1(t)$, $y_2(t)$… They are expressed by the following equations:

$$\begin{aligned}
&y_1(t) = \sum_i a_i(t) A_1 \cos 2\pi f(t-\tau_i(t)) \\
&y_2(t) = \sum_i a_i(t) B_1 \cos\left(2\pi f(t-\tau_i(t)) - \frac{\pi}{2}\right) = \sum_i a_i(t) B_1 \sin 2\pi f(t-\tau_i(t)) \\
&y_3(t) = \sum_i a_i(t) A_2 \cos 4\pi f(t-\tau_i(t)) \\
&y_4(t) = \sum_i a_i(t) B_2 \cos\left(4\pi f(t-\tau_i(t)) - \frac{\pi}{2}\right) = \sum_i a_i(t) B_2 \sin 4\pi f(t-\tau_i(t)) \\
&\dots ,
\end{aligned} \tag{4}$$

where factors A_i and B_i, $i = 1,2, \dots$, represent amplitudes of the corresponding sinusoidal signals (cos function without a phase shift and with phase shift $\pi/2$) for the subsequent harmonic frequencies.

The resulting received signal takes the form

$$\begin{aligned}
&y(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t) + \dots = \\
&= \sum_i a_i(t) A_1 \cos 2\pi f(t-\tau_i(t)) + \sum_i a_i(t) B_1 \sin 2\pi f(t-\tau_i(t)) + \\
&+ \sum_i a_i(t) A_2 \cos 4\pi f(t-\tau_i(t)) + \sum_i a_i(t) B_2 \sin 4\pi f(t-\tau_i(t)) + \dots \ .
\end{aligned} \tag{5}$$

Further note that (5) can be written as

$$y(t) = \sum_i a_i(t) \sum_{n=1}^{\infty} \left(A_n \cos 2\pi fn(t-\tau_i(t)) + B_n \sin 2\pi fn(t-\tau_i(t)) \right) . \tag{6}$$

As any periodic signal $x(t) = x(t + 1/f_0)$ that meets the Dirichlet conditions can be decomposed into a Fourier series defined with the following formula [Szabatin 2016]

$$x(t) = A_0 + \left(\sum_{n=1}^{\infty} A_n \cos 2\pi f_0 nt + B_n \sin 2\pi f_0 nt \right), \tag{7a}$$

$$x(t-\tau_i(t)) = A_0 + \sum\nolimits_{n=1}^{\infty}\left(A_n \cos 2\pi f_0 n\left(t-\tau_i(t)\right) + B_n \sin 2\pi f_0 n\left(t-\tau_i(t)\right)\right), \qquad (7b)$$

where f_0 means the fundamental frequency, while A_0 is the constant component in the above trigonometric series, and by substituting (7b) into (6) and assuming that $A_0 = 0$, the result is

$$y(t) = \sum_i a_i(t) x(t-\tau_i(t)) \ . \qquad (8)$$

Note: in the above derivation, frequency f is identified with frequency f_0.

Now observe that formula (8) is identical to formula (2). It is therefore demonstrated that the latter is indeed correct for periodic signals. It now only needs to be demonstrated further that it is also valid for aperiodic, i.e. any, signals. This issue is analysed in the two following sections.

A note on the assumption $A_0 = 0$ made above: due to the band selectivity of radio signals used in radio communications, the assumption is fully valid.

3. REPRESENTATION OF APERIODIC SIGNALS

In Section 2 it was demonstrated that formula (2) is fulfilled for periodic signals. In order for it to be valid for any signal, its correctness for aperiodic signals needs to be demonstrated as well. We use the derivations presented in Section 2. We will also take advantage of approximation, with the required accuracy, of signals in any form using a sum of sinusoidal signals with frequencies that are multiples of a certain fundamental frequency, i.e. using a Fourier series.

In this section we will handle the latter issue in greater detail. Let us try to find a connection between periodic and aperiodic signal representation based on the assumption available in [Landman et al. 2008], i.e. find an answer to the question if aperiodic signals can, under certain conditions, be treated as if they were periodic? This would enable us to apply the reasoning from Section 2 to aperiodic signal analysis and to generalise formula (2).

The analysis presented below will identify three signals related to each other: $x_0(t)$, $x_1(t)$ and $x_2(t)$.

Let us begin with signal $x_1(t)$. This is how any aperiodic signal will be denoted. As in [Szabatin 2016], it can be analysed using the Fourier transform defined as follows:

$$x_1(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df, \qquad (9a)$$

$$X(f) = \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt \quad , \tag{9b}$$

where $X(f)$ is an image within the frequency domain of signal $x_1(t)$.

Let signal $x_0(t)$ mean a signal created on the basis of the signal $x_1(t)$ derived previously by cutting (windowing) a fragment of length T. This can therefore be analytically represented as follows:

$$x_0(t) = \begin{cases} x_1(t) & \text{for} \quad t \in \langle 0, T \rangle \\ 0 & \text{for} \quad t > T \end{cases} \quad . \tag{10}$$

Finally, let signal $x_2(t)$ be a periodic signal created by a periodic repetition of signal $x_0(t)$ with period T.

To better visualise the relations between signals $x_1(t)$, $x_0(t)$ and $x_2(t)$, they are illustrated in Fig. 1.

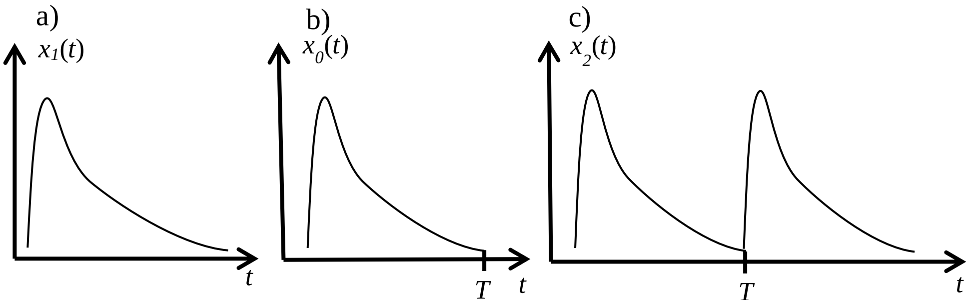

Fig. 1. Visualization of relations between the signals $x_1(t)$, $x_0(t)$ and $x_2(t)$
Source: own study.

Figure 1a shows a sample aperiodic signal; Figure 1b shows a fragment with length T, "cut" from that signal, while the signal shown in Fig. 1c can be referred to as the "periodic version" of signal $x_1(t)$. Fig. 1c shows that after time T, the main section of this signal (i.e. $x_0(t)$) is cyclically repeated, and additionally it can be stated that $x_1(t) \approx x_2(t) = x_2(t+T)$, when T is sufficiently long.

Signal $x_2(t)$, as a periodic signal, is represented using a Fourier series, in this case in a complex form [Szabatin 2016]

$$x_2(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} \quad , \tag{11}$$

where c_n are coefficients of the complex Fourier series, calculated as follows:

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_2(t) e^{-j2\pi n f_0 t} dt, \tag{12}$$

where T is the period of the periodic signal $x_2(t)$, f_0 is the frequency corresponding to this period, and n defines subsequent harmonic constituents. The following equation stems from the above:

$$f_0 = \frac{1}{T} \ . \tag{13}$$

Let us now present signal $x_2(t)$ in a different form, substituting (12) into (11):

$$x_2(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_2(t) e^{-j2\pi n f_0 t} dt \right) e^{j2\pi n f_0 t} \ . \tag{14}$$

Signal $x_0(t)$, as it is not periodic, has no representation in the form of a Fourier series along the entire time line, but only for moments from a time window of duration T, where it is identical to signal $x_2(t)$: it can be represented there with a Fourier series for signal $x_2(t)$.

Let us now imagine that signal $x_1(t)$ is really periodic, but its period T approaches infinity: $T \to \infty$ (its repetition appears after an infinitely long time – this assumption has already been made in [Landman et al. 2008]). What happens when we assume that signal $x_2(t)$ becomes similar to $x_1(t)$, when $T \to \infty$? Let us investigate what happens in this case, when T is sufficiently long, i.e. , where T_d should be understood as the length of the window applied to signal $x_1(t)$, which forms signal $x_0(t)$, such that it encompasses 95% of the energy of signal $x_1(t)$. Let us create an auxiliary function (signal) $B(t)$, which can be identified with any representative of signal $x_2(t)$, whose period fulfils the above conditions. We can write it as follows:

$$B(t) = x_2(t)\Big|_{T \gg T_d} \ , \tag{15a}$$

As we will show later in this section, definition (15a) is also valid for the boundary case, i.e. for $T \to \infty$. In other words, function

$$B(t) = x_2(t)\Big|_{T \to \infty} , \tag{15b}$$

is defined correctly and does exist.

Note: The above statement leads to the conclusion that $B(t)$ functions here in dual roles: as a family of functions with the property stated above, and as a single member of this family. In the deliberations below, we will only use it for representatives defined in equation (15b).

Substituting (14) into (15b), we receive

$$B(t)=\sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x_2(t)e^{-j2\pi nf_0t}dt\right)e^{j2\pi nf_0t}\Bigg|_{T\to\infty}. \tag{16}$$

Note now that due to the above property of signal $x_1(t)$, which is a signal of finite energy, the mean signal energy $x_2(t)$ in period T decreases with increasing length of this period. Therefore, for all sufficiently high values of T, i.e. for $T>>T_d$, one can state

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}x_2(t)e^{-j2\pi nf_0t}dt\cong\int_{-\frac{T_d}{2}}^{\frac{T_d}{2}}x_2(t)e^{-j2\pi nf_0t}dt\,. \tag{17}$$

Furthermore, since $x_2(t)$ is a periodic version of $x_1(t)$ since relations between signals $x_1(t)$, $x_0(t)$ and $x_2(t)$ are as defined above, the following relations are valid:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}x_2(t)e^{-j2\pi nf_0t}dt\cong\int_{-\infty}^{\infty}x_0(t)e^{-j2\pi nf_0t}dt\cong\int_{-\infty}^{\infty}x_1(t)e^{-j2\pi nf_0t}dt \tag{18}$$

for sufficiently high values of period T.

To illustrate the effect of period T length on the accuracy achieved in equation (18), let us use Figure 2.

For a certain finite T, only a portion of signal $x_1(t)$ is repeated, limited by T. However, if we imagine T as infinitely long, it can be assumed that the "entirety" of signal $x_1(t)$ is repeated. The longer T is, the greater is the accuracy of the approximation $x_1(t)\approx x_2(t)$ along the entire time line within the section from 0 to T.

On account of the above, by substituting (18) into (16), we obtain

$$B(t)=\sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_{-\infty}^{\infty}x_1(t)e^{-j2\pi nf_0t}dt\right)e^{j2\pi nf_0t}\Bigg|_{T\to\infty}. \tag{19}$$

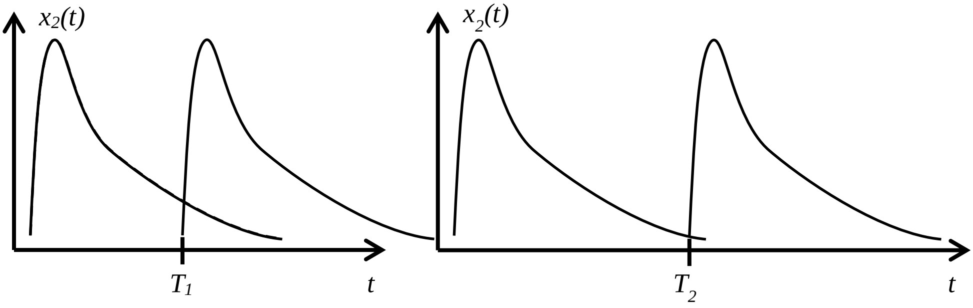

Fig. 2. Impact of extending period T on the shape of function x2(t) and the accuracy of the approximations in (18)

Furthermore, the longer T is, the more the fundamental frequency of the harmonics decreases, frequency spectral lines concentrate, and the distances between them approach zero: the spectrum transforms from discrete ($f_0 n$ for $n = 1,2,3\ldots$) into continuous (f domain, as in (9b)). Therefore, for sufficiently high T values, let us introduce a new variable in (19) to denote nf_0:

$$f_0 n \to f \;. \tag{20}$$

Then (19) takes the form

$$B(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T} \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt \right) e^{j2\pi ft} \Bigg|_{T\to\infty} . \tag{21}$$

Now, using the relation (9b) in (21), we can replace a section of equation (21) with a transform of signal $x_1(t)$:

$$B(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} X(f) e^{j2\pi nft} \Bigg|_{T\to\infty} . \tag{22}$$

Let us note further that (13) means that if $T \to \infty$, then $f_0 \to 0$, thus we can state:

$$\Delta f = (n+1) f_0 - nf_0 = f_0 \to 0 \;. \tag{23}$$

This allows us to rewrite (22) as

$$B(t) = \sum_{n=-\infty}^{\infty} X(f) e^{j2\pi ft} \Delta f \Bigg|_{\Delta f\to 0} . \tag{24}$$

Note now that the sum form in (24) contains the definition of Riemann integer. Using this fact and comparing (24) with (9a), we can state

$$B(t) = \int_{-\infty}^{\infty} X\left(f\right) e^{j2\pi ft} df = x_1\left(t\right) \quad . \tag{25}$$

We can therefore see that indeed, if $T \to \infty$, then the signal spectrum obtained using a decomposition into a Fourier series becomes convergent on the spectrum obtained using the Fourier transform.

The validity of the above derivations can be verified using the MATLAB environment. This was used to simulate an aperiodic signal $x_1(t)$: in this case, it was a signal comprising one triangular signal period, where the descending part of which lasted twice as long as the ascending part. The charts of this signal is shown in Fig. 3.

This signal can be defined as follows:

$$x_1\left(t\right) = \begin{cases} \dfrac{A}{L} t & \text{for} \quad 0 \le t \le L \\ -\dfrac{A}{2L} t + \dfrac{3A}{2} & \text{for} \quad L < t \le 3L \\ 0 & \text{for} \quad t > 3L \end{cases} . \tag{26}$$

where A is signal amplitude (in this case A = 5), and L is duration of the triangular signal ascending slope (L = 5 s).

In the next step it was assumed that this signal would be repeated after time T. Coefficients A_n and B_n of the Fourier series of the periodic signal $x_2(t)$ created in this way were calculated, and on their basis a chart of the signal reconstructed from the k harmonics ($k = 10$ was assumed) was drawn. The reconstruction was performed for multiple T values, both for those for which only part of the original signal was repeated (Fig. 3b – 10 s, Fig. 3c – 12.5 s), and for those for which the entirety of the signal was processed (Fig. 3d – 15 s, Fig. 3e – 20 s).

In analysing Figs. 3b-3, it can be observed that increasing T does indeed result in the reconstructed signal being increasingly similar to the original. In this case, only when all of the signal's energy is repeated (period T is at least equal to signal duration $x_1(t)$, i.e. 15 s), the reconstructed signal slopes are properly inclined and the signal takes values in the correct range from 0 to A (i.e. to 5). Additionally, signal repetition, absent from the original signal after all, occurs at increasingly later points. Fig. 3 graphically visualises the correctness of formula (25).

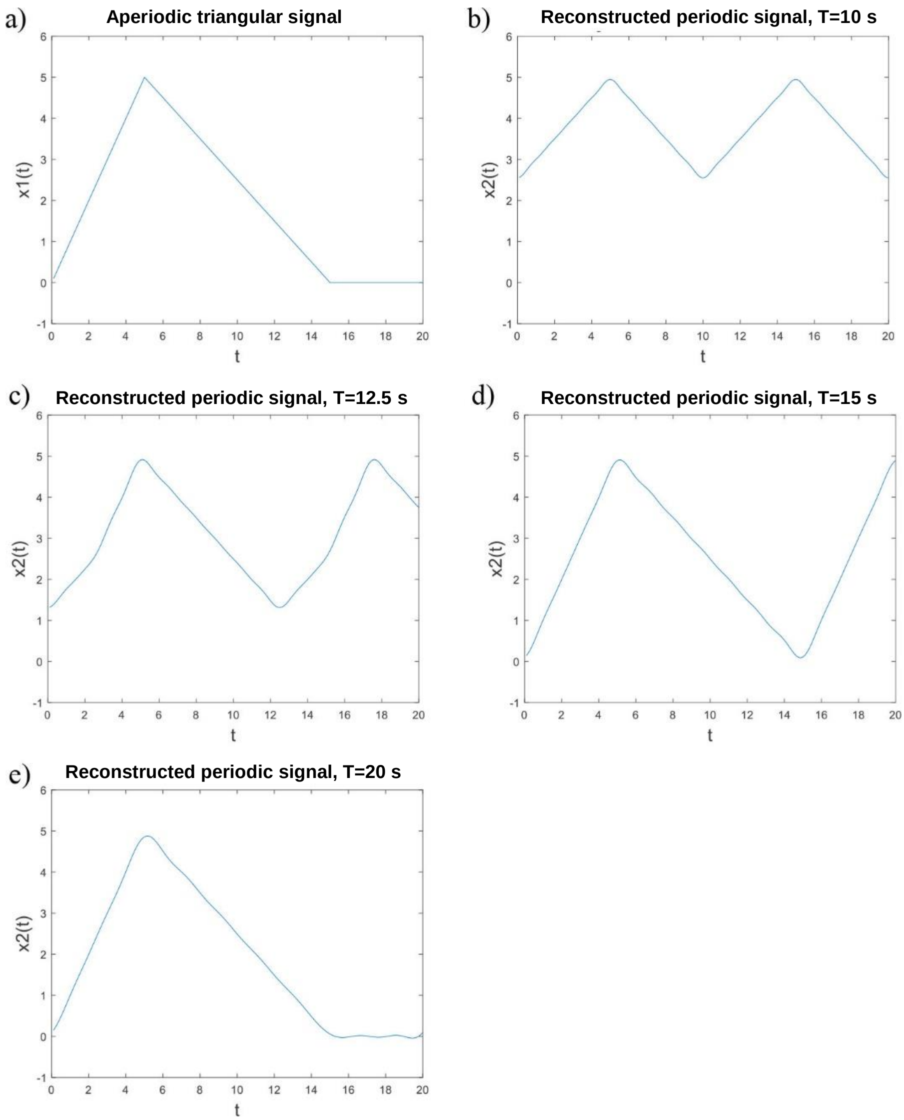

Fig. 3. Effect of extending period T on the shape of function x2(t) in MATLAB: a) original signal; b) reconstructed signal, T = 10 s; c) reconstructed signal, T = 12.5 s; d) reconstructed signal, T = 15 s; e) reconstructed signal, T = 20 s

4. DERIVATION OF INPUT-OUTPUT RELATIONS FOR APERIODIC SIGNALS

The deliberations in Section 3 prove that aperiodic signals can indeed be treated as periodic signals whose period approaches infinity. Section 2 demonstrated, on the other hand, that if a radio channel input signal is a periodic signal, the principle of superposition can be used to sum the component signals (which we have identified with Fourier series components) and consequently present the output signal of this channel in a generalised form (2). This section combines all the above reasoning, as aperiodic signals can (under certain conditions) be treated as periodic ones. We will attempt to decompose them into their component signals on the basis of a Fourier series (remembering the assumption concerning the period of such signals), and then, using the superposition principle, to sum the assumed constituents to obtain the output signal.

As in Section 2, we also assume a signal, $y(t)$, which is, in accordance with the superposition principle, a sum of individual signals $y_1(t)$, $y_2(t)$… $y_n(t)$ originating from point sources generating sinusoidal signals (components of the Fourier series). This time, however, let us assume that the period of the individual sinusoidal signals approaches infinity, which means that the fundamental frequency is close to zero and therefore the difference Δf between individual harmonics ($\Delta f = f_2 - f_1$, etc.) is minimal as well. Therefore:

$$\left.\begin{aligned}
&y_1(t) = \sum_i a_i(t) A_1 \cos 2\pi f_1 (t - \tau_i(t)) \\
&y_2(t) = \sum_i a_i(t) B_1 \sin 2\pi f_1 (t - \tau_i(t)) \\
&y_3(t) = \sum_i a_i(t) A_2 \cos 2\pi f_2 (t - \tau_i(t)) = \sum_i a_i(t) A_2 \cos 4\pi f_1 (t - \tau_i(t)) \\
&y_4(t) = \sum_i a_i(t) B_2 \sin 2\pi f_2 (t - \tau_i(t)) = \sum_i a_i(t) B_2 \sin 4\pi f_1 (t - \tau_i(t)) \\
&\ldots ,
\end{aligned}\right|_{\Delta f \to 0} \tag{27}$$

where, as in Section 2, A_i and B_i, $i = 1,2,\ldots$, represent amplitudes of the corresponding sinusoidal constituents.

By summing these we achieve, as in (5), a contraction into a Fourier series:

$$\begin{aligned}
y(t) &= y_1(t) + y_2(t) + y_3(t) + y_4(t) + \ldots = \\
&= \sum_i a_i(t) \sum_{n=1}^{\infty} \left(A_n \cos 2\pi f_1 n (t - \tau_i(t)) + B_n \sin 2\pi f_1 n (t - \tau_i(t)) \right)\Big|_{\Delta f \to 0} \quad .
\end{aligned} \tag{28}$$

The Fourier series presented in (28) is a trigonometric series. In the derivations in this section we would like to obtain a formula similar to (2), where any aperiodic signal would occur as $x(t- \tau_i(t))$, such as in (9a). Since in (3) we used a Fourier series in a complex, non-trigonometric form, we also move to this form as well in these deliberations. By using Euler's equations, we can represent $\cos\varphi$ and $\sin\varphi$ as:

$$\cos\varphi = \frac{e^{j\varphi}+e^{-j\varphi}}{2}, \quad \sin\varphi = \frac{e^{j\varphi}-e^{-j\varphi}}{2j}, \tag{29}$$

therefore, by substituting (29) into (28), and assuming that

$$\varphi = 2\pi f_1(t-\tau_i(t)), \tag{30}$$

we achieve:

$$\begin{aligned} y(t) &= \sum_i a_i(t)\sum_{n=1}^{\infty}\left(A_n\frac{e^{jn\varphi}+e^{-jn\varphi}}{2} + B_n\frac{e^{jn\varphi}-e^{-jn\varphi}}{2j}\right)\Big|_{\Delta f\to 0} = \\ &= \sum_i a_i(t)\sum_{n=1}^{\infty}\left(\left(\frac{A_n}{2}+\frac{B_n}{2j}\right)e^{jn\varphi} + \left(\frac{A_n}{2}-\frac{B_n}{2j}\right)e^{-jn\varphi}\right)\Big|_{\Delta f\to 0} = \\ &= \sum_i a_i(t)\left(\sum_{n=1}^{\infty}\left(\frac{A_n-jB_n}{2}\right)e^{jn\varphi} + \sum_{n=1}^{\infty}\left(\frac{A_n+jB_n}{2}\right)e^{-jn\varphi}\right)\Big|_{\Delta f\to 0} \quad . \end{aligned} \tag{31}$$

Since according to [Szabatin 2016] the following relations exist:

$$c_{-n} = c_n^*; \quad c_n = \frac{A_n - jB_n}{2} \tag{32}$$

and by assuming, as in Section 2, that $A_0 = 0$ (i.e. $c_0 = 0$), we achieve

$$\begin{aligned} y(t) &= \sum_i a_i(t)\left(\sum_{n=1}^{\infty} c_n e^{jn\varphi} + \sum_{n=1}^{\infty} c_n^* e^{-jn\varphi}\right)\Big|_{\Delta f\to 0} = \\ &= \sum_i a_i(t)\left(\sum_{n=1}^{\infty} c_n e^{jn\varphi} + \sum_{n=-\infty}^{-1} c_n^* e^{jn\varphi}\right)\Big|_{\Delta f\to 0} = \\ &= \sum_i a_i(t)\sum_{n=-\infty}^{\infty} c_n e^{jn\varphi}\Bigg|_{\substack{c_0=0,\\ \Delta f\to 0}} = \sum_i a_i(t)\sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_1 n(t-\tau_i(t))}\Bigg|_{\substack{c_0=0,\\ \Delta f\to 0}} \quad . \end{aligned} \tag{33}$$

The final section of the formula achieves the desired Fourier series in a complex form.

When discussing (20) it was explained that the concentration of spectral lines (i.e. $\Delta f \rightarrow 0$) is a result of $T \rightarrow \infty$, which incidentally is one of the assumptions concerning component signals, mentioned at the beginning of this section. We can therefore replace the previous conditions

$$y(t) = \sum_i a_i(t) \sum_{n=-\infty}^{\infty} c_n e^{j2\pi fn(t-\tau_i(t))} \Big|_{T\rightarrow\infty} \quad . \tag{34}$$

If we recall that according to (11), any periodic signal ($x_2(t)$) can be defined as

$$x_2\left(t-\tau_i(t)\right) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nf_1\left(t-\tau_i(t)\right)}, \tag{35}$$

and furthermore in (25) and all of Section 3 we demonstrated that an aperiodic signal ($x_1(t)$) can be treated as periodic under the condition that its period approaches infinity, i.e.

$$x_1\left(t\right) = x_2\left(t\right)\Big|_{T\rightarrow\infty}, \tag{36}$$

then by substituting (35) and (36) into (34), we can finally state that

$$y(t) = \sum_i a_i(t) x_2\left(t-\tau_i(t)\right)\Big|_{T\rightarrow\infty} = \sum_i a_i(t) x_1\left(t-\tau_i(t)\right). \tag{37}$$

We have thus again obtained formula (2) as we intended, this time for aperiodic signals.

5. SUMMARY

The paper includes an analysis of the linear model of a radio channel with time-dependent parameters, taking into account the multipath propagation phenomenon. The paper generalises the relation known in the literature, which connects input signals with the outputs for this type of channel. Previously its validity had only been demonstrated for the case when the input signal is a sinusoidal signal. It was shown in this paper that, by using the superposition principle and the relations between the Fourier series and transform, the relation discussed above is true for both periodic and aperiodic signals, and is therefore valid for any signal.

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